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# Parameters estimation for GARCH (p,q) model: QL and AQL approaches

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# Parameters estimation for GARCH (p,q) model: QL and AQL approaches

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In this paper, estimation for the generalized autoregressive conditional heteroscedasticity (GARCH) model is conducted. The Quasi likelihood (QL) and Asymptotic Quasi-likelihood (AQL) estimation methods are suggested in this paper. The QL approach relaxes the distributional assumptions of GARCH processes. The AQL technique obtains out the QL method when the conditional variance of process is unknown. The AQL methodology, merging the kernel technique used for parameter estimation of the GARCH model. This AQL methodology enables a substitute technique for parameter estimation when the conditional variance of process is unknown. Application of the QL and AQL methods to weekly prices changes of crude oil modelled by GARCH model is considered.

**keywords:** GARCH model; Quasi likelihood (QL); Asymptotic Quasi-likelihood (AQL); Kernel estimator, Crude oil prices.

## 1 Introduction

The generalized autoregressive conditional heteroscedasticity GARCH(p,q) process $y_t$ is defined by

$$y_t = \mu + \xi_t, \qquad t = 1, 2, 3, \cdots, T. \tag{1.1}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1\xi_{t-1}^2 + \cdots + \alpha_p\xi_{t-p}^2 + \beta_1\sigma_{t-1}^2 + \cdots + \beta_q\sigma_{t-q}^2 + \zeta_t, \qquad t = 1, 2, 3, \cdots, T. \tag{1.2}$$

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$\xi_t$ are i.i.d with $E(\xi_t) = 0$ and $V(\xi_t) = \sigma_t^2$; and $\zeta_t$ are i.i.d with $E(\zeta_t) = 0$ and $V(\zeta_t) = \sigma_\zeta^2$. The GARCH model are developed by Bollerslev (1986) to extend the earlier work on ARCH models by Engle (1982). For estimation and applications of (GARCH) models ( See, Bollerslev et al. (1992); Engle (2001); Diebold and Lopez (1995); Pagan (1996); Palm (1996); Andersen and Bollerslev (1998); Engle and Patton (2001) and Andersen et al. (2006)). Moreover, GARCH models have now become standard textbook material in econometrics and finance as exemplified by, e.g., Alexander (2001), Enders (2004), and Taylor (2004).

Weiss (1986) and Bollerslev and Wooldridge (1992) obtained Quasi-maximum likelihood (QML) estimator to GARCH models. They also shows that the estimators of the parameters obtained by maximizing a likelihood function constructed under the normality assumption can still be consistent even if the true density is not normal. In many cases, there is evidence that the standardized residuals from estimated GARCH models are not normally distributed, especially for high-frequency financial time. Anyfantaki and Demos (2011) suggests employing a Markov chain Monte Carlo algorithm which allows the calculation of a classical estimator via the simulated EM algorithm. They also outline the issues that the recursive nature of the conditional variance makes exact likelihood analysis of these models computationally infeasible. Moreover, for semi-parametric and nonparametric estimation of the GARCH models (see, Linton and Yan (2011); Yang (2006); Linton et al. (2010); Jianqing et al. (2014)).

Existing techniques for parameter estimation in GARCH models are mainly maximum likelihood based. This means that the probability structure of $\{y_t\}$ has to be known. Usually it assume $\{y_t\}$ has conditional Gaussian distribution. This concern is very valid in finance as empirical data reveal fat-tailness and skewness which contradicts to the conditional normality. Therefore, it might lead estimation procedure to be exposed to modelling errors.

This paper applies the Quasi-likelihood (QL) and Asymptotic Quasi-likelihood (AQL) approaches to (GARCH) model. The QL approach relaxes the distributional assumptions but has a restriction that assumes the conditional variance process is known. To overcome this limitation, we suggest a substitute technique, the AQL methodology, merging the kernel technique used for parameter estimation of the GARCH model. This AQL methodology enables a substitute technique for parameter estimation when the conditional variance of process is unknown.

This paper is structured as follows. The QL and AQL approaches are introduced and the GARCH model estimation using the QL and AQL methods are developed in Section 2. Reports of simulation outcomes, and numerical cases are presented in Section 3. The QL and AQL techniques are applied to weekly prices changes of crude oil modeled by GARCH in Section 4. Section 5 summarizes and concludes the paper.

# 2 Parameter estimation of GARCH(p,q) model using the QL and AQL methods

In the following, parameter estimation for GARCH(p,q) model, which include non-linear and non-Gaussian models is given. We propose QL and AQL approaches for estimation of GARCH(p,q) model. The estimations of unknown parameters are considered without any distribution assumptions concerning the processes involved and the estimation is based on different scenarios in which the conditional covariance of the error's terms are assumed to be known or unknown.

## 2.1 The QL method

Let the observation equation be given by

$$\mathbf{y}_t = \mathbf{f}_t(\theta) + \zeta_t, \qquad t = 1, 2, 3 \cdots, T, \tag{2.1.1}$$

$\zeta_t$ is a sequence of martingale difference with respect to $\mathcal{F}_t$, $\mathcal{F}_t$ denotes the $\sigma$-field generated by $\mathbf{y}_t, \mathbf{y}_{t-1}, \cdots, \mathbf{y}_1$ for $t \geq 1$; that is, $E(\zeta_t|\mathcal{F}_{t-1}) = E_{t-1}(\zeta_t) = 0$; where $\mathbf{f}_t(\theta)$ is an $\mathcal{F}_{t-1}$ measurable; and $\theta$ is parameter vector, which belongs to an open subset $\Theta \in R^d$. Note that $\theta$ is a parameter of interest. We assume that $E_{t-1}(\zeta_t \zeta_t{}') = \mathbf{\Sigma}_t$ is known. Now, the liner class $\mathcal{G}_T$ of the estimating function (EF) can be defined by

$$\mathcal{G}_T = \{\sum_{t=1}^{T} \mathbf{W}_t(\mathbf{y}_t - \mathbf{f}_t(\theta))\}$$

and the quasi-likelihood estimation function (QLEF) can be defined by

$$\mathbf{G}_T^*(\theta) = \sum_{t=1}^{T} \dot{\mathbf{f}}_t(\theta) \mathbf{\Sigma}_t^{-1}(\mathbf{y}_t - \mathbf{f}_t(\theta)) \tag{2.1.2}$$

where $\mathbf{W}_t$ is $\mathcal{F}_{t-1}$-measureable and $\dot{\mathbf{f}}_t(\theta) = \partial \mathbf{f}_t(\theta)/\partial\theta$. Then, the estimation of $\theta$ by the QL method is the solution of the QL equation $\mathbf{G}_T^*(\theta) = 0$ (see Hedye (1997)).

If the sub-estimating function spaces of $\mathcal{G}_T$ are considered as follows,

$$\mathcal{G}_t = \{\mathbf{W}_t(\mathbf{y}_t - \mathbf{f}_t(\theta))\}$$

then the QLEF can be defined by

$$\mathbf{G}_{(t)}^*(\theta) = \dot{\mathbf{f}}_t(\theta) \mathbf{\Sigma}_t^{-1}(\mathbf{y}_t - \mathbf{f}_t(\theta)) \tag{2.1.3}$$

and the estimation of $\theta$ by the QL method is the solution of the QL equation $\mathbf{G}_{(t)}^*(\theta) = 0$.

A limitation of the QL method is that the nature of $\mathbf{\Sigma}_t$ may not be obtainable. A misidentified $\mathbf{\Sigma}_t$ could result in a deceptive inference about parameter $\theta$. In the next subsection, we introduce the AQL method, which is basically the QL estimation assuming that the covariance matrix $\mathbf{\Sigma}_t$ is unknown.

## 2.2 The AQL method

The QLEF (see 2.1.2 and 2.1.3) relies on the information of $\Sigma_t$. Such information is not always accessible. To find the QL when $E_{t-1}(\zeta_t {\zeta_t}')$ is not accessible, Lin (2000) proposed the AQL method.
Definition 2.2.1: Let $\mathbf{G}^*_{T,n}$ be a sequence of the EF in $\mathcal{G}$. For all $\mathbf{G}_T \in \mathcal{G}$, if

$$(E\dot{\mathbf{G}}_T)^{-1}(E\mathbf{G}_T\mathbf{G}_T)'(E\dot{\mathbf{G}}'_T)^{-1} - (E\dot{\mathbf{G}}^*_{T,n})^{-1}(E\mathbf{G}^*_{T,n}\mathbf{G}^{*\prime}_T)(E\dot{\mathbf{G}}^{*\prime}_{T,n})^{-1}$$

is asymptotically non-negative definite, $\mathbf{G}^*_{T,n}$ can be denoted as the asymptotic quasi-likelihood estimation function (AQLEF) sequence in $\mathcal{G}$, and the AQL sequence estimates $\theta_{T,n}$ by the AQL method is the solution of the AQL equation $\mathbf{G}^*_{T,n} = 0$.

Suppose, in probability, $\mathbf{\Sigma}_{t,n}$ is converging to $E_{t-1}(\zeta_t {\zeta_t}')$. Then,

$$\mathbf{G}^*_{T,n}(\theta) = \sum_{t=1}^{T} \dot{\mathbf{f}}_t(\theta)\mathbf{\Sigma}^{-1}_{t,n}(\mathbf{y}_t - \mathbf{f}_t(\theta)) \tag{2.2.1}$$

expresses an AQLEF sequence. The solution of $\mathbf{G}^*_{T,n}(\theta) = 0$ expresses the AQL sequence estimate $\{\theta^*_{T,n}\}$, which converges to $\theta$ under certain regular conditions.

In this paper, the kernel smoothing estimator of $\mathbf{\Sigma}_t$ is suggested to find $\mathbf{\Sigma}_{t,n}$ in the AQLEF (2.2.1). A wide-ranging appraisal of the Nadaraya–Watson (NW) estimator-type kernel estimator is available in Härdle (1990) and Wand and Jones (1995). By using these kernel estimators, the AQL equation becomes

$$\mathbf{G}^*_{T,n}(\theta) = \sum_{t=1}^{T} \dot{\mathbf{f}}_t(\theta)\hat{\mathbf{\Sigma}}^{-1}_{t,n}(\hat{\theta}^{(0)})(\mathbf{y}_t - \mathbf{f}_t(\theta)) = 0. \tag{2.2.2}$$

The estimation of $\theta$ by the AQL method is the solution to (2.2.2). Iterative techniques are suggested to solve the AQL equation (2.2.2). Such techniques start with the ordinary least squares (OLS) estimator $\hat{\theta}^{(0)}$ and use $\hat{\mathbf{\Sigma}}_{t,n}(\hat{\theta}^{(0)})$ in the AQL equation (2.2.2) to obtain the AQL estimator $\hat{\theta}^{(1)}$. Repeat this a few times until it converges.

The next subsections present the parameter estimation of GARCH model using the QL and AQL methods.

## 2.3 Parameter estimation of GARCH(p,q) model using the QL method

The GARCH(p,q) process is defined by

$$y_t = \mu + \xi_t, \qquad t = 1, 2, 3, \cdots, T. \tag{2.3.1}$$

and

$$\sigma^2_t = \alpha_0 + \alpha_1\xi^2_{t-1} + \cdots + \alpha_p\xi^2_{t-p} + \beta_1\sigma^2_{t-1} + \cdots + \beta_q\sigma^2_{t-q} + \zeta_t, \qquad t = 1, 2, 3, \cdots, T. \tag{2.3.2}$$

$\xi_t$ are i.i.d with $E_{t-1}(\xi_t) = 0$ and $V_{t-1}(\xi_t) = \sigma^2_t$; and $\zeta_t$ are i.i.d with $E_{t-1}(\zeta_t) = 0$ and $V_{t-1}(\zeta_t) = \sigma^2_\zeta$. For this scenario, the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \cdots - \alpha_p \xi_{t-p}^2 - \beta_1 \sigma_{t-1}^2 - \cdots - \beta_q \sigma_{t-q}^2 \end{pmatrix}.$$

The (QLEF), to estimate $\sigma_t^2$, is given by

$$G_{(t)}(\sigma_t^2) = (0,1) \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1} \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}$$

$$= \sigma_\zeta^{-2}(\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \cdots - \alpha_p \xi_{t-p}^2 - \beta_1 \sigma_{t-1}^2 - \cdots - \beta_q \sigma_{t-q}^2). \tag{2.3.3}$$

Given $\hat{\xi}_0 = 0$, initial values $\psi_0 = (\mu_0, \alpha_{0_0}, \alpha_{1_0}, \cdots, \alpha_{p_0}, \beta_{1_0}, \cdots, \beta_{q_0}, \sigma_{\zeta_0}^2)$, $\hat{\xi}_{t-i}^2 = (y_{t-i} - \mu_0)^2$, and $\hat{\sigma}_{t-j}^2$ is the QL estimation of $\sigma_{t-j}^2$ where i=1,2,$\cdots$,p and j=1,2,$\cdots$,q, then the QL estimation of $\sigma_t^2$ is the solution of $G_{(t)}(\sigma_t^2) = 0$,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \cdots + \alpha_p \hat{\xi}_{t-p}^2 + \beta_1 \hat{\sigma}_{t-1}^2 + \cdots + \beta_q \hat{\sigma}_{t-q}^2, \qquad t = 1, 2, 3 \cdots, T. \tag{2.3.4}$$

The QLEF, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, to estimate the parameters $\theta = \mu$, $\alpha_0$, $\alpha_1$, $\cdots$, $\alpha_q$, $\beta_1$, $\cdots$, $\beta_q$ is given by

$$G_T(\theta) = \sum_{t=1}^{T} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\xi_{t-p}^2 \\ 0 & -\sigma_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\sigma_{t-q}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\zeta_0}^2 \end{pmatrix}^{-1} \begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}$$

The QL estimate of $\theta = (\mu, \alpha_0, \alpha_1, \cdots, \alpha_q, \beta_1, \cdots, \beta_q)$ is the solution of $G_T(\theta) = 0$. where $\hat{\zeta}_t = \hat{\sigma}_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1 \hat{\xi}_{t-1}^2 - \cdots - \hat{\alpha}_p \hat{\xi}_{t-p}^2 - \hat{\beta}_1 \hat{\sigma}_{t-1}^2 - \cdots - \hat{\beta}_q \hat{\sigma}_{t-q}^2, t = 1, 2, 3, \cdots, T$ and

$$\hat{\sigma}_\zeta^2 = \frac{\sum_{t=1}^{T} (\hat{\zeta}_t - \bar{\hat{\zeta}})^2}{T - 1} \tag{2.3.5}$$

$\hat{\psi} = (\hat{\mu}, \hat{\alpha}_0, \hat{\alpha}_1, \cdots, \hat{\alpha}_p, \hat{\beta}_1, \cdots, \hat{\beta}_q, \hat{\sigma_\zeta^2})$ is an initial value in the iterative procedure.

## 2.4 Parameter estimation of GARCH(p,q) model using the AQL method

For GARCH(p,q) model given by (2.3.1) and (2.3.2) and using the same argument listed under (2.3.2). Firstly, to estimate $\sigma_t^2$, so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma_t^2) = (0,1)\mathbf{\Sigma}_{t,n}^{-1}\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}$$

Given $\hat{\xi_0} = 0$, $\theta_0 = (\mu_0, \alpha_{0_0}, \alpha_{1_0}, \cdots, \alpha_{p_0}, \beta_{1_0}, \cdots, \beta_{q_0})$, $\mathbf{\Sigma}_{t,n}^{(0)} = \mathbf{I}_2$, and $\hat{\xi}_{t-i}^2 = (y_{t-i} - \mu_0)^2$, and $\hat{\sigma}_{t-j}^2$ is the AQL estimation of $\sigma_{t-j}^2$ where i=1,2,$\cdots$,p and j=1,2,$\cdots$,q, then the AQL estimation of $\sigma_t^2$ is the solution of $G_{(t)}(\sigma_t^2) = 0$, that is,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1\hat{\xi}_{t-1}^2 + \cdots + \alpha_p\hat{\xi}_{t-p}^2 + \beta_1\hat{\sigma}_{t-1}^2 - \cdots - \beta_q\hat{\sigma}_{t-q}^2, \qquad t = 1,2,3\cdots,T. \quad (2.4.1)$$

Secondly, by kernel estimation method, we find

$$\hat{\mathbf{\Sigma}}_{t,n}(\theta^{(0)}) = \begin{pmatrix} \hat{\sigma}_n(y_t) & \hat{\sigma}_n(y_t, \sigma_t) \\ \hat{\sigma}_n(\sigma_t, y_t) & \hat{\sigma}_n(\sigma_t) \end{pmatrix}.$$

Thirdly, to estimate the parameters $\theta_0 = (\mu_0, \alpha_0, \alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q)$, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$ and the sequence of (AQLEF)

$$G_T(\theta) = \sum_{t=1}^{T}\begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\xi_{t-p}^2 \\ 0 & -\sigma_{t-1}^2 \\ \vdots & \vdots \\ 0 & -\sigma_{t-q}^2 \end{pmatrix}\hat{\mathbf{\Sigma}}_{t,n}^{-1}\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix}.$$

The AQL estimate of $\theta = (\mu, \alpha_0, \alpha_1, \cdots, \alpha_q, \beta_1, \cdots, \beta_q)$ is the solution of $G_T(\theta) = 0$. The estimation procedure will be iteratively repeated until it converges.

# 3 Simulation study

In this section we report results from simulation studies which design to evaluate the empirical performance of the proposed QL and AQL approaches for parameter estimation. One specific example of model (1.1) and (1.2) are considered in the simulation, which is related to a heteroscedastic model GARCH(1,1)

$$y_t = \mu + \xi_t, \qquad t = 1,2,3,\cdots,T. \quad (3.1)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1\xi_{t-1}^2 + \beta_1\sigma_{t-1} + \zeta_t, \qquad t = 1,2,3,\cdots,T. \quad (3.2)$$

$\xi_t$ are i.i.d with $E_{t-1}(\xi_t) = 0$ and $V_{t-1}(\xi_t) = \sigma_t^2$; and $\zeta_t$ are i.i.d with $E_{t-1}(\zeta_t) = 0$ and $V_{t-1}(\zeta_t) = \sigma_\zeta^2$.

### 3.1 Parameter estimation of GARCH(1,1) model using the QL method

For GARCH(1,1) given by (3.1) and (3.2), the martingale difference is

$$\begin{pmatrix} \xi_t \\ \zeta_t \end{pmatrix} = \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \beta_1 \sigma_{t-1}^2 \end{pmatrix}.$$

The (QLEF), to estimate $\sigma_t^2$, is given by

$$\begin{aligned} G_{(t)}(\sigma_t^2) &= (0,1) \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_\zeta^2 \end{pmatrix}^{-1} \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \beta_1 \sigma_{t-1}^2 \end{pmatrix} \\ &= \sigma_\zeta^{-2}(\sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \beta_1 \sigma_{t-1}^2). \end{aligned} \tag{3.1.1}$$

Given $\hat{\xi_0} = 0$, initial values $\psi_0 = (\mu_0, \alpha_{0_0}, \alpha_{1_0}, \beta_{1_0}, \sigma_{\zeta_0}^2)$, $\hat{\xi}_{t-1}^2 = (y_{t-1} - \mu_0)^2$, and $\hat{\sigma}_{t-1}^2$ is the QL estimation of $\sigma_{t-1}^2$, then the QL estimation of $\sigma_t^2$ is the solution of $G_{(t)}(\sigma_t^2) = 0$,

$$\hat{\sigma}_t^2 = \alpha_0 + \alpha_1 \hat{\xi}_{t-1}^2 + \beta_1 \hat{\sigma}_{t-1}^2, \qquad t = 1, 2, 3 \cdots, T. \tag{3.1.2}$$

To estimate the parameters $\mu$, $\alpha_0$, $\alpha_1$ and $\beta_1$, using $\{\hat{\sigma}_t^2\}$ and $\{y_t\}$, The QLEF is given by

$$G_T(\mu, \alpha_0, \alpha_1, \beta_1) = \sum_{t=1}^{T} \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & -\xi_{t-1}^2 \\ 0 & -\sigma_{t-1}^2 \end{pmatrix} \begin{pmatrix} \sigma_t^2 & 0 \\ 0 & \sigma_{\zeta_0}^2 \end{pmatrix}^{-1}$$

$$* \begin{pmatrix} y_t - \mu \\ \sigma_t^2 - \alpha_0 - \alpha_1 \xi_{t-1}^2 - \beta_1 \sigma_{t-1}^2 \end{pmatrix}.$$

The solution of $G_T(\mu, \alpha_0, \alpha_1, \beta_1) = 0$ is the QL estimate of $\mu$, $\alpha_0$, $\alpha_1$ and $\beta_1$. Therefore

$$\hat{\mu} = \sum_{t=1}^{T} \frac{y_t}{\hat{\sigma}_t^2} \Big/ \sum_{t=1}^{T} \frac{1}{\hat{\sigma}_t^2}. \tag{3.1.3}$$

$$\hat{\beta}_1 = \frac{S_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2} S_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} - S_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2} S_{\hat{\sigma}_t^2 \hat{\sigma}_{t-1}^2}}{S_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2}^2 - S_{\hat{\sigma}_{t-1}^2 \hat{\sigma}_{t-1}^2} S_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2}}. \tag{3.1.4}$$

$$\hat{\alpha}_1 = \frac{S_{\hat{\sigma}_t^2 \hat{\xi}_{t-1}^2} - \hat{\beta}_1 S_{\hat{\sigma}_{t-1}^2 \hat{\xi}_{t-1}^2}}{S_{\hat{\xi}_{t-1}^2 \hat{\xi}_{t-1}^2}}. \tag{3.1.4}$$

$$\hat{\alpha}_0 = \frac{\sum_{t=1}^{T} \hat{\sigma}_t^2 - \hat{\alpha}_1 \sum_{t=1}^{T} \hat{\xi}_{t-1}^2 - \hat{\beta}_1 \sum_{t=1}^{T} \hat{\sigma}_{t-1}^2}{T}. \tag{3.1.5}$$

and let

$$\hat{\sigma}^2_\zeta = \frac{\sum_{t=1}^{T}(\hat{\zeta}_t - \bar{\hat{\zeta}})^2}{T-1} \tag{3.1.6}$$

where

- $\hat{\zeta}_t = \hat{\sigma}^2_t - \hat{\alpha}_0 - \hat{\alpha}_1\hat{\xi}^2_{t-1} - \hat{\beta}_1\hat{\sigma}^2_{t-1}, \qquad t = 1, 2, 3, \cdots, T,$
- $S_{\hat{\sigma}^2_{t-1}\hat{\xi}^2_{t-1}} = \sum_{t=1}^{T}\hat{\sigma}^2_{t-1}\hat{\xi}^2_{t-1} - \frac{\sum_{t=1}^{T}\hat{\sigma}^2_{t-1}\sum_{t=1}^{T}\hat{\xi}^2_{t-1}}{T},$
- $S_{\hat{\sigma}^2_t\hat{\xi}^2_{t-1}} = \sum_{t=1}^{T}\hat{\sigma}^2_t\hat{\xi}^2_{t-1} - \frac{\sum_{t=1}^{T}\hat{\sigma}^2_t\sum_{t=1}^{T}\hat{\xi}^2_{t-1}}{T},$
- $S_{\hat{\xi}^2_{t-1}\hat{\xi}^2_{t-1}} = \sum_{t=1}^{T}\hat{\xi}^4_{t-1} - \frac{\left(\sum_{t=1}^{T}\hat{\xi}^2_{t-1}\right)^2}{T},$
- $S_{\hat{\sigma}^2_t\hat{\sigma}^2_{t-1}} = \sum_{t=1}^{T}\hat{\sigma}^2_t\hat{\sigma}^2_{t-1} - \frac{\sum_{t=1}^{T}\hat{\sigma}^2_t\sum_{t=1}^{T}\hat{\sigma}^2_{t-1}}{T},$
- $S_{\hat{\sigma}^2_{t-1}\hat{\sigma}^2_{t-1}} = \sum_{t=1}^{T}\hat{\sigma}^4_{t-1} - \frac{\left(\sum_{t=1}^{T}\hat{\sigma}^2_{t-1}\right)^2}{T}.$

$\hat{\psi} = (\hat{\mu}, \hat{\alpha}_0, \hat{\alpha}_1, \hat{\beta}_1, \hat{\sigma^2_\zeta})$ is an initial value in the iterative procedure.
The initial values might be affected the estimation results. For extensive discussion on assigning initial values in the (QL) estimation procedures (see Alzghool and Lin (2008, 2011), Alzghool (2016) and Alzghool and Al-Zubi (2016)).

## 3.2 Parameter estimation of GARCH(1,1) model using the AQL method

For GARCH(1,1) model given by (3.1) and (3.2) and using the same argument listed under (3.1) and (3.2). Firstly, to estimate $\sigma^2_t$, so the sequence of (AQLEF) is given by

$$G_{(t)}(\sigma^2_t) = (0,1)\mathbf{\Sigma}^{-1}_{t,n}\begin{pmatrix} y_t - \mu \\ \sigma^2_t - \alpha_0 - \alpha_1\xi^2_{t-1} - \beta_1\sigma^2_{t-1}\end{pmatrix}$$

Given $\hat{\xi_0} = 0$, $\theta_0 = (\mu_0, \alpha_{0_0}, \alpha_{1_0}, \beta_{1_0})$, $\mathbf{\Sigma}^{(0)}_{t,n} = \mathbf{I}_2$, $\hat{\xi}^2_{t-1} = (y_{t-1} - \mu_0)^2$ and $\hat{\sigma}^2_{t-1}$ is the AQL estimation of $\sigma^2_{t-1}$, then the AQL estimation of $\sigma^2_t$ is the solution of $G_{(t)}(\sigma^2_t) = 0$, that is,

$$\hat{\sigma}^2_t = \alpha_0 + \alpha_1\hat{\xi}^2_{t-1} + \beta_1\hat{\sigma}^2_{t-1}, \qquad t = 1, 2, 3\cdots, T. \tag{3.2.1}$$

Secondly, by kernel estimation method, we find

$$\hat{\mathbf{\Sigma}}_{t,n}(\theta^{(0)}) = \begin{pmatrix} \hat{\sigma}_n(y_t) & 0 \\ 0 & \hat{\sigma}_n(\sigma_t)\end{pmatrix}.$$

Thirdly, to estimate the parameters $\theta = (\mu, \alpha_0, \alpha_1, \beta_1)$, using $\{\hat{\sigma}^2_t\}$ and $\{y_t\}$ and the sequence of (AQLEF)

$$G_T(\mu,\alpha_0,\alpha_1,\beta_1)=\sum_{t=1}^{T}\begin{pmatrix}-1 & 0\\ 0 & -1\\ 0 & -\hat{\xi}_{t-1}^2\\ 0 & -\hat{\sigma}_{t-1}^2\end{pmatrix}\hat{\mathbf{\Sigma}}_{t,n}^{-1}\begin{pmatrix} y_t-\mu \\ \sigma_t^2-\alpha_0-\alpha_1\xi_{t-1}^2-\beta_1\sigma_{t-1}^2\end{pmatrix}.$$

The AQL estimate of $\mu$, $\alpha_0$, $\alpha_1$, and $\beta_1$ is the solution of $G_T(\mu,\alpha_0,\alpha_1,\beta_1)=0$. Therefore

$$\hat{\mu}=\sum_{t=1}^{T}\frac{y_t}{\hat{\sigma}_n(y_t)}\Big/\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(y_t)}. \tag{3.2.2}$$

$$\hat{\beta}_1=\frac{SS_{\hat{\sigma}_{t-1}^2\hat{\xi}_{t-1}^2}SS_{\hat{\sigma}_t^2\hat{\xi}_{t-1}^2}-SS_{\hat{\xi}_{t-1}^2\hat{\xi}_{t-1}^2}SS_{\hat{\sigma}_t^2\hat{\sigma}_{t-1}^2}}{SS^2_{\hat{\sigma}_{t-1}^2\hat{\xi}_{t-1}^2}-SS_{\hat{\sigma}_{t-1}^2\hat{\sigma}_{t-1}^2}SS_{\hat{\xi}_{t-1}^2\hat{\xi}_{t-1}^2}}. \tag{3.2.3}$$

$$\hat{\alpha}_1=\frac{SS_{\hat{\sigma}_t^2\hat{\xi}_{t-1}^2}-\hat{\beta}_1SS_{\hat{\sigma}_{t-1}^2\hat{\xi}_{t-1}^2}}{SS_{\hat{\xi}_{t-1}^2\hat{\xi}_{t-1}^2}}. \tag{3.2.4}$$

$$\hat{\alpha}_0=\frac{\sum_{t=1}^{T}\frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)}-\hat{\alpha}_1\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}-\hat{\beta}_1\sum_{t=1}^{T}\frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}}{\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)}}, \tag{3.2.5}$$

and let

$$\hat{\sigma}_\zeta^2=\frac{\sum_{t=1}^{T}(\hat{\zeta}_t-\bar{\hat{\zeta}})^2}{T-1} \tag{3.2.6}$$

where

- $\hat{\zeta}_t=\hat{\sigma}_t^2-\hat{\alpha}_0-\hat{\alpha}_1\hat{\xi}_{t-1}^2-\hat{\beta}_1\hat{\sigma}_{t-1}^2, \qquad t=1,2,3,\cdots,T,$
- $SS_{\hat{\sigma}_{t-1}^2\hat{\xi}_{t-1}^2}=(\sum_{t=1}^{T}\frac{\hat{\sigma}_{t-1}^2\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)})(\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)})-(\sum_{t=1}^{T}\frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)})(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}),$
- $SS_{\hat{\sigma}_t^2\hat{\xi}_{t-1}^2}=(\sum_{t=1}^{T}\frac{\hat{\sigma}_t^2\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)})(\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)})-(\sum_{t=1}^{T}\frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)})(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}),$
- $SS_{\hat{\xi}_{t-1}^2\hat{\xi}_{t-1}^2}=(\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)})(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^4}{\hat{\sigma}_n(\sigma_t)})-(\sum_{t=1}^{T}\frac{\hat{\xi}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)})^2,$
- $SS_{\hat{\sigma}_t^2\hat{\sigma}_{t-1}^2}=(\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)})\sum_{t=1}^{T}\frac{\hat{\sigma}_t^2\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)}-\sum_{t=1}^{T}\frac{\hat{\sigma}_t^2}{\hat{\sigma}_n(\sigma_t)}\sum_{t=1}^{T}\frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)},$
- $SS_{\hat{\sigma}_{t-1}^2\hat{\sigma}_{t-1}^2}=(\sum_{t=1}^{T}\frac{1}{\hat{\sigma}_n(\sigma_t)})(\sum_{t=1}^{T}\frac{\hat{\sigma}_{t-1}^4}{\hat{\sigma}_n(\sigma_t)})-(\sum_{t=1}^{T}\frac{\hat{\sigma}_{t-1}^2}{\hat{\sigma}_n(\sigma_t)})^2.$

Table 1: The QL and AQL estimates and The RMSE of each estimates is stated below that estimate.

| | $\mu$ | $\alpha_0$ | $\alpha_1$ | $\beta_1$ | $\mu$ | $\alpha_0$ | $\alpha_1$ | $\beta_1$ |
|---|---|---|---|---|---|---|---|---|
| true | 0.15 | 0.65 | 0.87 | 0.10 | 0.20 | 0.41 | 0.88 | 0.08 |
| QL | 0.149 | 0.779 | 0.865 | 0.074 | 0.199 | 0.461 | 0.912 | 0.057 |
| | 0.040 | 0.353 | 0.011 | 0.029 | 0.031 | 0.155 | 0.033 | 0.025 |
| AQL | 0.150 | 0.661 | 0.851 | 0.092 | 0.209 | 0.405 | 0.901 | 0.076 |
| | 0.001 | 0.012 | 0.019 | 0.009 | 0.010 | 0.006 | 0.021 | 0.004 |
| true | -0.10 | 0.48 | 0.89 | 0.08 | 0.16 | 0.37 | 0.9 | 0.08 |
| QL | -0.101 | 0.556 | 0.902 | 0.058 | 0.159 | 0.434 | 0.922 | 0.058 |
| | 0.034 | 0.212 | 0.014 | 0.024 | 0.030 | 0.189 | 0.024 | 0.025 |
| AQL | -0.110 | 0.486 | 0.891 | 0.0752 | 0.161 | 0.374 | 0.911 | 0.076 |
| | 0.010 | 0.006 | 0.001 | 0.005 | 0.001 | 0.004 | 0.011 | 0.004 |
| true | 0.18 | 0.39 | 0.88 | 0.08 | 0.09 | 0.50 | 0.89 | 0.05 |
| QL | 0.179 | 0.447 | 0.892 | 0.058 | 0.089 | 0.538 | 0.898 | 0.036 |
| | 0.031 | 0.146 | 0.015 | 0.024 | 0.033 | 0.090 | 0.009 | 0.015 |
| AQL | 0.180 | 0.395 | 0.882 | 0.076 | 0.091 | 0.504 | 0.892 | 0.046 |
| | 0.001 | 0.005 | 0.002 | 0.005 | 0.002 | 0.004 | 0.002 | 0.004 |

The estimation procedure will be iteratively repeated until it converges.

For this simulation study, samples of size T = 500 are taken, and the mean and root mean squared errors (RMSE) for $\hat{\mu}$, $\hat{\alpha}_0$, $\hat{\alpha}_1$, and $\hat{\beta}_1$ are calculated, where N = 1000 independent samples. In Table 1, QL represents the QL estimate and AQL represents the AQL estimate. The effect of the sample size on the estimation of parameters is considered. Samples of sizes T = 20, 40, 60, 80, and 100 were generated. In Table 2, The results are revealed that the RMSE will be decreases when the sample size is increase.

# 4 Application to GARCH model

The QL and AQL methods developed in earlier section apply to real-life data where the data are modeled by GARCH model (1.1) and (1.2). The data set contains the weekly price changes of Crude oil prices $P_t$. The $P_t$ of Cushing, OK West Texas Intermediate (US Dollars per Barrel) for period from 7/1/2000 to 10/6/2016, 858 observations in total. The data are obtained from the US Energy Information Administration (see, http://www.eia.gov/dnav/pet). $P_t$ appear not to be stationary, as indicated in Fig. **??**.

Table 2: The QL and AQL estimates and The RMSE of each estimates is stated below that estimate.

| | | $\mu$ | $\alpha_0$ | $\alpha_1$ | $\beta_1$ | $\mu$ | $\alpha_0$ | $\alpha_1$ | $\beta_1$ |
|---|---|---|---|---|---|---|---|---|---|
| | true | 0.16 | 0.37 | 0.90 | 0.08 | -0.10 | 0.48 | 0.89 | 0.08 |
| T=20 | QL | 0.17 | 0.42 | 0.89 | 0.07 | -0.09 | 0.51 | 0.90 | 0.06 |
| | | 0.176 | 0.511 | 0.008 | 0.016 | 0.169 | 0.451 | 0.018 | 0.022 |
| | AQL | 0.16 | 0.38 | 0.89 | 0.07 | -0.10 | 0.47 | 0.90 | 0.07 |
| | | 0.037 | 0.012 | 0.007 | 0.014 | 0.066 | 0.014 | 0.013 | 0.018 |
| T=40 | QL | 0.16 | 0.42 | 0.89 | 0.07 | -0.09 | 0.51 | 0.91 | 0.06 |
| | | 0.149 | 0.422 | 0.007 | 0.016 | 0.137 | 0.326 | 0.018 | 0.021 |
| | AQL | 0.16 | 0.38 | 0.89 | 0.07 | -0.10 | 0.47 | 0.90 | 0.07 |
| | | 0.027 | 0.012 | 0.007 | 0.013 | 0.022 | 0.014 | 0.012 | 0.016 |
| T=60 | QL | 0.16 | 0.42 | 0.89 | 0.07 | -0.09 | 0.52 | 0.91 | 0.06 |
| | | 0.121 | 0.289 | 0.007 | 0.018 | 0.119 | 0.307 | 0.018 | 0.021 |
| | AQL | 0.16 | 0.38 | 0.89 | 0.07 | -0.10 | 0.47 | 0.90 | 0.07 |
| | | 0.019 | 0.012 | 0.007 | 0.011 | 0.014 | 0.013 | 0.012 | 0.015 |
| T=80 | QL | 0.16 | 0.42 | 0.89 | 0.07 | -0.10 | 0.51 | 0.90 | 0.06 |
| | | 0.100 | 0.159 | 0.007 | 0.017 | 0.108 | 0.248 | 0.018 | 0.021 |
| | AQL | 0.16 | 0.38 | 0.89 | 0.07 | -0.10 | 0.47 | 0.90 | 0.07 |
| | | 0.012 | 0.012 | 0.007 | 0.011 | 0.012 | 0.013 | 0.012 | 0.015 |
| T=100 | QL | 0.16 | 0.42 | 0.89 | 0.07 | -0.10 | 0.51 | 0.90 | 0.06 |
| | | 0.100 | 0.159 | 0.007 | 0.018 | 0.101 | 0.242 | 0.018 | 0.021 |
| | AQL | 0.16 | 0.38 | 0.89 | 0.07 | -0.10 | 0.47 | 0.90 | 0.07 |
| | | 0.012 | 0.011 | 0.007 | 0.011 | 0.011 | 0.013 | 0.012 | 0.015 |

The data are transformed into rates of change by taking the first difference of the logs. Thus, $y_t = log(P_t) - log(P_{t-1})$. The series of $y_t$ is presented in Fig. **??** and fit $\{y_t\}$ by using GARCH (1,1):

$$y_t = \mu + \xi_t, \qquad t = 1, 2, 3, \cdots, T. \tag{3.1}$$

and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \xi_{t-1}^2 + \beta_1 \sigma_{t-1} + \zeta_t, \qquad t = 1, 2, 3, \cdots, T. \tag{3.2}$$

$\xi_t$ are i.i.d with $E_{t-1}(\xi_t) = 0$ and $V_{t-1}(\xi_t) = \sigma_t^2$; and $\zeta_t$ are i.i.d with $E_{t-1}(\zeta_t) = 0$ and $V_{t-1}(\zeta_t) = \sigma_\zeta^2$.

Table 3: Estimation of $\mu, \alpha_0, \alpha_1, \beta_1$ for the rates of change prices data

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| | $\hat{\mu}_0$ | $\hat{\alpha}_0$ | $\hat{\alpha}_1$ | $\hat{\beta}_1$ | $\frac{\bar{\hat{\xi}}_t}{S.d(\hat{\xi}_t)}$ |
|---|---|---|---|---|---|
| QL | 0.0008 | 0.566 | 0.912 | 0.0004 | 0.002 |
| AQL | 0.0089 | 0.630 | 0.972 | 0.041 | 0.185 |

Table 3 Indications the estimates of $\mu$, $\alpha_0$, $\alpha_1$, and $\beta_1$ achieves by two methods. QL represents the estimate found by QL method, AQL represents the asymptotic quasi-likelihood estimate.

We can see from the fourth column in Table 3 that QL gives smaller standardized residuals. The QL method tends to be more efficient than AQL method.

## 5 Summary

In this paper, the estimation of the parameters in GARCH models has been presented by two alternative approaches. The article has shown that the QL and AQL estimating procedures are provided an efficient approach for estimating the unknown parameter when the exactly probability structure of underlying model is unknown. It will provide a robust tool for obtaining optimal point estimate of parameters in heteroscedastic models, like GARCH model.

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